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In memory of S.I. Pohozaev

Classical Solutions of the Vlasov–Poisson Equations with External Magnetic Field in a Half-Space[1]

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Abstract—We consider the first mixed problem for the Vlasov–Poisson equations with an external magnetic field in a half-space. This problem describes the evolution of the density distributions of ions and electrons in a high temperature plasma with a fixed potential of electric field on a boundary. For arbitrary potential of electric field and sufficiently large induction of external magnetic field, it is shown that the characteristics of the Vlasov equations do not reach the boundary of the halfspace. It is proved the existence and uniqueness of classical solution with the supports of charged-particle density distributions at some distance from the boundary, if initial density distributions are sufficiently small.

Keywords: Vlasov–Poisson equations, mixed problem, classical solutions, external magnetic field, half-space.

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1. INTRODUCTION

The Vlasov equations were first obtained in [1]. Now they are one of the best-known mathematical models in the kinetic theory of gases including high temperature plasma. The study of these equations has made it possible to predict a number of physical phenomena such as Landau damping effect [2]. For the applications of Vlasov equations in physics, see [3] and the bibliography given there.

The majority of mathematical papers in the field of Vlasov equations are devoted to the Cauchy problem. The global solvability of the "smoothed" Vlasov equations was investigated in the papers of Braun and Hepp [4], Maslov [5], and Dobrushin [6].

The existence of a global generalized solution of the Cauchy problem for the Vlasov–Poisson equations was proved by Arsen'ev [7]. The existence of a global generalized solution and its weak stability in the case of the Cauchy problem for the Vlasov–Poisson and the Vlasov–Maxwell equations were studied in [8–12].

The existence and uniqueness of a classical solution of the Cauchy problem for the Vlasov–Poisson system with small time or small initial data was proved by Arsen'ev [13] and Bardos and Degond [14]. Global classical solutions of the initial problem for the Vlasov–Poisson equations were studied in [15–21]. The Landau damping effect was studied in [22, 23].

Much less attention has been paid to the existence of solutions of the Vlasov equations in domains with boundary. The studies here have been mostly focused on generalized solutions of mixed problems for the Vlasov–Poisson equations and the Vlasov–Maxwell equations, see Arsen'ev [24], Weckler [25], and others. The global existence of classical solutions of mixed problems for the Vlasov–Poisson equations in a half-space with Neumann or Dirichlet boundary conditions for the electric-field potential and the conditions of elastic reflection for charged-particle density distributions on the boundary was proved by Guo [26], Hwang, and Velázquez [27]. The main difficulties in the study of classical solutions for these prob-

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lems have to do with the behaviour of the characteristics near the boundary. The existence of classical and strong solutions of mixed problems in the general case is still an open problem, see Kozlov [11], Samarskii [28], and Weckler [25]. This problem is relevant to the design of a controlled thermonuclear fusion reactor, a mathematical model of which is described by mixed problems for the Vlasov system with respect to the density distributions of charged particles of opposite signs in a bounded domain. The production of a stable high-temperature plasma in a reactor requires that the so-called plasma column be strictly inside the domain during some time interval in order to keep it away from the vacuum container wall [3, 29].

In this paper we consider the Vlasov–Poisson system of equations

$$-\Delta\varphi(x,t) = 4\pi e\int_{\mathbb{R}^3}\sum_{\beta=\pm1}\beta f^{\beta}(x,p,t)dp \quad (x\in\mathbb{R}^3_+,\ 0<t<T), \tag{1.1}$$

$$\frac{\partial f^{\beta}}{\partial t} + \frac{1}{m^{\beta}}(p,\nabla_x f^{\beta}) + \beta e\Big(-\nabla_x\varphi + \frac{1}{m^{\beta}c}[p,B],\nabla_p f^{\beta}\Big) = 0$$
$$(x\in\mathbb{R}^3_+,\ p\in\mathbb{R}^3,\ 0<t<T,\ \beta=\pm1). \tag{1.2}$$

Here $\varphi=\varphi(x,t)$ is the potential of the self-consistent electric field, $f^{\beta}=f^{\beta}(x,p,t)$ is the density distribution function of positively charged ions (for $\beta=+1$) or of electrons (for $\beta=-1$) at a point x with impulse p at a time t, ∇_x and ∇_p are the gradients with respect to x and p, respectively, m^{+1} and m^{-1} are the masses of an ion and electron, e is the electron charge, c is the velocity of light, $B=B(x)$ is the induction of an external magnetic field, $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^3$, $[\cdot,\cdot]$ is the vector product in $\mathbb{R}^3$, and $\mathbb{R}^3_+=\{x=(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1>0\}$.

Assume that the following initial conditions hold:

$$f^{\beta}(x,p,t)|_{t=0} = f_0^{\beta}(x,p) \quad (x\in\overline{\mathbb{R}^3_+},\ p\in\mathbb{R}^3,\ \beta=\pm1), \tag{1.3}$$

where f_0^{β} are given nonnegative functions.

Suppose also that the function $\varphi(x,t)$ satisfies the Dirichlet condition

$$\varphi(x,t)|_{x_1=0} = 0 \quad (x'=(x_2,x_3)\in\mathbb{R}^2,\ 0\le t<T). \tag{1.4}$$

Unlike other authors, we consider the solutions having supports of the charged-particle density distributions, which lie at some distance from the boundary. To obtain such a solution we assume first that in some layer the external magnetic field B is parallel to the boundary $\{x\in\mathbb{R}^3 : x_1=0\}$ and is sufficiently strong, and second that the initial density distributions $f_0^{\beta}(x,p)$ have compact supports lying at some distance from the boundary. These assumptions imply that the characteristics do not intersect the boundary. This phenomenon can be interpreted physically as follows: the charged particles do not reach the walls of the vacuum chamber of the thermonuclear fusion reactor because they move along trajectories close to the Larmor ones. According to [3], the presence of a considerable number of particles on the boundary can result in either destruction of the reactor walls or in cooling of the high-temperature plasma due to its contact with the reactor walls. In majority of thermonuclear fusion reactors an external magnetic field is used as a control ensuring plasma confinement [3, 29]. As distinct from other papers (see, for example [27]), which have dealt with the Vlasov–Poisson equations for particles of the same sign, we are concerned here with those equations for a two-component plasma, since the word "plasma" is used in physics to designate this high-temperature state of an ionized gas with charge neutrality [3].

The paper is organized as follows. In Section 2 we introduce the notation and formulate the assumptions concerning the external magnetic field $B(x)$ and the initial density distributions $f_0^{\beta}(x,p)$. In Section 3 we study the characteristics of system (1.2) in $\mathbb{R}^3_+\times\mathbb{R}^3\times(0,T)$ for a fixed potential φ. If the magnetic field B is sufficiently strong, we prove that the characteristics starting from some layer $\{x\in\mathbb{R}^3: 5\delta/8\le x_1\le\delta\}$, $\delta>0$, do not reach the boundary $\{x\in\mathbb{R}^3 : x_1=0\}$. This phenomenon is well known in plasma physics. Charged particles are moving along trajectories, similar to circular or helical paths (so-called, Larmor trajectories) with sufficiently small amplitudes. In Section 4 we consider estimates of characteristics derivatives with respect to initial data and dynamics of supports for charged-particle density distributions. Section 5 is devoted to the estimates of norms for functions $F_\varphi(x,t) = \int_{\mathbb{R}^3} \sum_{\beta=\pm1} \beta f_\varphi^\beta(x,p,t)dp$, where f_φ^β is a solution of problem (1.2), (1.3) with a fixed potential of electric field φ. In Section 6 for sufficiently small initial density distribution functions we prove the existence and uniqueness of a classical solution to problem (1.1)–(1.4) with the supports, which lie at some distance from the boundary. We note that the Vlasov–Poisson equations for a two-component plasma with external magnetic field in a half-space and in a cylinder were studied in [30–32]. In contrast to these papers we obtain explicit upper bound for the norms of initial density distributions providing the existence and uniqueness of a classical solution to problem (1.1)–(1.4).

2. NOTATION. THE MAIN RESULT

2.1. Denote by $C^s(\mathbb{R}^n)$ $(C^s(\overline{\mathbb{R}^n_+}))$, $s \geq 0$, $n \in \mathbb{N}$, the Hölder space of continuous functions in $\mathbb{R}^n$ $(\overline{\mathbb{R}^n_+})$, having continuous derivatives in $\mathbb{R}^n$ $(\overline{\mathbb{R}^n_+})$ up to the kth order, $k = [s]$, with the finite norm

$$\begin{gathered} \|u\|_s = \max_{|\alpha|\leq k} \sup_x |\mathcal{D}^\alpha u(x)| \quad \text{for} \quad s = k \in \mathbb{Z}, \quad 0 \leq k, \\ \|u\|_s = \|u\|_k + |u|_{k+\sigma} \quad \text{for} \quad s = k+\sigma, \quad 0 \leq k \in \mathbb{Z}, \quad 0 < \sigma < 1, \end{gathered} \tag{2.1}$$

where

$$|u|_{k+\sigma} = \max_{|\alpha|=k} \sup_{x\neq y} |x-y|^{-\sigma} |\mathcal{D}^\alpha u(x) - \mathcal{D}^\alpha u(y)|, \tag{2.2}$$

$$\mathcal{D}^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n),$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad \mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \colon x_1 > 0\}.$$

Let $C(\mathbb{R}^n) = C^0(\mathbb{R}^n)$ and $C(\overline{\mathbb{R}^n_+}) = C^0(\overline{\mathbb{R}^n_+})$.

Similarly we can define the space $C^1(\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T])$ of bounded continuous functions with bounded continuous first order derivatives in $\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T]$.

Remark 2.1. If $s = k + \sigma$, $0 \leq k \in \mathbb{Z}$, and $0 < \sigma < 1$, then we can endow $C^s(\mathbb{R}^n)$ (respectively, $C^s(\overline{\mathbb{R}^n_+})$) with the equivalent norm

$$\|u\|'_s = \|u\|_k + |u|_{k+\sigma,b}, \tag{2.3}$$

where

$$|u|_{k+\sigma,b} = \max_{|\alpha|=k} \sup_{\substack{x\neq y, \\ |x-y|<b}} |x-y|^{-\sigma} |\mathcal{D}^\alpha u(x) - \mathcal{D}^\alpha u(y)|, \quad 0 < b \leq 1, \tag{2.4}$$

and $\|u\|'_s \leq \|u\|_s \leq k_1 \|u\|'_s$, $k_1 = k_1(\sigma) > 0$ does not depend on u.

Remark 2.2. For any $s \geq 0$, the spaces $C^s(\mathbb{R}^n)$ and $C^s(\overline{\mathbb{R}^n_+})$ are Banach spaces. If $s = k + \sigma$, $0 \leq k \in \mathbb{Z}$, and $0 < \sigma < 1$, then the space $C^s(\mathbb{R}^n)$ $(C^s(\overline{\mathbb{R}^n_+}))$ is not separable, and the set of infinitely differentiable functions in $\mathbb{R}^n (\overline{\mathbb{R}^n_+})$ with finite norm $\|\cdot\|_s$ is not dense in $C^s(\mathbb{R}^n)$ $(C^s(\overline{\mathbb{R}^n_+}))$.

Let $\dot{C}^k(\mathbb{R}^n)$ with $k, n \in \mathbb{N}$ denote the space of k-times continuously differentiable functions on $\mathbb{R}^n$ having compact supports.

Denote by $\hat{C}^s(\overline{\mathbb{R}^3_+})$ the space of vector-valued functions $Y = (Y_1, Y_2, Y_3)$ having coordinates $Y_i \in C^s(\overline{\mathbb{R}^3_+})$ with the norm

$$\|Y\|_s = \max_{0\le m\le k}\langle Y\rangle_m, \quad \langle Y\rangle_m = \left\{\sum_{i=1}^{3}\max_{|\alpha|=m}\|\mathcal{D}^\alpha Y_i\|_0^2\right\}^{\frac{1}{2}} \quad \text{for} \quad s = k\in\mathbb{Z},\ 0\le k,$$

$$\|Y\|_s = \|Y\|_k + \langle Y\rangle_{k+\sigma}, \quad \langle Y\rangle_{k+\sigma} = \left\{\sum_{i=1}^{3}|Y_i|_{k+\sigma}^2\right\}^{\frac{1}{2}} \quad \text{for} \quad s = k+\sigma, \tag{2.5}$$
$$0\le k\in\mathbb{Z}, \quad 0<\sigma<1.$$

We introduce the Banach space $C([0,T], C^s(\overline{\mathbb{R}^3_+}))$, $s>0$, of continuous functions $[0,T]\ni t\mapsto \varphi(\cdot,t)\in C^s(\overline{\mathbb{R}^3_+})$ with the norm

$$\|\varphi\|_{s,T} = \sup_{0\le t\le T}\|\varphi(\cdot,t)\|_s. \tag{2.6}$$

Similarly we can define the space $C([0,T], C^s_\Omega(\overline{\mathbb{R}^3_+}))$, where $C^s_\Omega(\overline{\mathbb{R}^3_+}) = \{w\in C^s(\overline{\mathbb{R}^3_+}) : \operatorname{supp} w\subset\bar{\Omega}\}$, $\Omega\subset\mathbb{R}^3_+$ is a bounded domain.

We also consider the Banach space $L_1((0,T), C^s(\overline{\mathbb{R}^3_+}))$, $s>0$, of the Lebesgue measurable functions $(0,T)\ni t\mapsto\varphi(\cdot,t)\in C^s(\overline{\mathbb{R}^3_+})$ with the norm

$$\|\varphi\|_{L_1((0,T),C^s(\overline{\mathbb{R}^3_+}))} = \int_0^T\|\varphi(\cdot,t)\|_s dt. \tag{2.7}$$

Let $M_{s,R} = \{\varphi\in C([0,T], C^s(\overline{\mathbb{R}^3_+})) : \|\varphi\|_{L_1((0,T),C^s(\overline{\mathbb{R}^3_+}))}\le R\}$, $R>0$, $s>0$. Clearly, $M_{s,R}$ is a complete metric space with metric $\rho_{s,R}(,\psi) = \|\varphi-\psi\|_{s,T}$.

Let $B_r(x_0) = \{x\in\mathbb{R}^3 : \left|x-x^0\right|<r\}$, $B_r = B_r(0)$, $|B_r| = 4\pi r^3/3$, and $R^3_\delta = \{x\in\mathbb{R}^3\colon 0<\delta<x_1\}$.

In what follows, k_i, c_j, $\hat{k}_i$, $\hat{c}_j$ are positive constants.

2.2. We now formulate the conditions which the magnetic field B and the initial charged-particle density distributions $f_0^\beta(x,p)$ must satisfy.

Condition 2.1. Let $B\in\hat{C}^{1+\sigma}(\overline{\mathbb{R}^3_+})$ and let $B(x) = (0,0,h)$ for $\delta/4\le x_1\le\varkappa$, where

$$\frac{16c}{e\delta}\psi(T)(\rho+\sqrt{3}eR) < h, \tag{2.8}$$

$\varkappa$, δ, ρ, R, $h>0$ do not depend on x, $9\delta/8<\min\{\varkappa,1\}$, and $\psi\in C([0,\infty),[0,1])$ is the nondecreasing function given by

$$\psi(t) = \begin{cases} 2^{-\frac{1}{2}}\left(1-\cos\left(\dfrac{eh}{m^{-1}c}t\right)\right)^{\frac{1}{2}}, & t\in\left[0,\dfrac{m^{-1}c\pi}{eh}\right], \\ 1, \quad t\in\left[0,\dfrac{m^{-1}c\pi}{eh},\infty\right). \end{cases}$$

Remark 2.3. Clearly, inequality (2.8) is equivalent to the following

$$3^{\frac{1}{2}}eR < \rho_1-\rho = \rho_2-\rho_1, \tag{2.9}$$

where ρ_1 and ρ_2 are such that

$$\rho_1 = \frac{he\delta}{16c\psi(T)}, \quad \rho_2 = 2\rho_1-\rho. \tag{2.10}$$

Denote

$$\mathcal{D}_0 = (\mathbb{R}^3_{\delta_0} \cap B_{\varkappa_0}) \times B_{\rho_0}, \tag{2.11}$$

where $\delta_0, \varkappa_0, \rho_0 > 0$ are such that $\delta < \delta_0 < \varkappa_0 < \varkappa - \delta/8$, $\rho_0 < \rho$.

Condition 2.2. Let $f_0^\beta \in \dot{C}^{1+\sigma}(\mathbb{R}^6)$, and let $\operatorname{supp} f_0^\beta \subset \mathcal{D}_0$.

Definition 2.1. A vector-valued function $\{\varphi, f^\beta\}$ with $\varphi \in C([0,T], C^{2+\sigma}(\overline{\mathbb{R}^3_+}))$ and $f^\beta \in C^1(\overline{\mathbb{R}^3_+} \times \mathbb{R}^3 \times [0,T])$ is called a classical solution of problem (1.1)–(1.4) if $\int_{\mathbb{R}^3} \beta f^\beta(\cdot, p, t) dp \in C([0,T], C^\sigma_\Omega(\overline{\mathbb{R}^3_+}))$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $\bar{\Omega} \subset \mathbb{R}^3_+$, $\varphi(x,t) \to 0$ as $|x| \to \infty$ for any $t \in [0,T]$, and $\{\varphi, f^\beta\}$ satisfies equations (1.1), (1.2), the initial conditions (1.3) and boundary condition (1.4).

3. TRAJECTORIES OF CHARGED PARTICLES IN MAGNETIC FIELD

3.1. Assume that Conditions 2.1 and 2.2 are satisfied. Given a fixed function $\varphi \in M_{2+\sigma,R}$, Eq. (1.2) with initial condition (1.3) can be solved using the method of characteristics. To this end, we consider the following system of ordinary differential equations:

$$\frac{dX_\varphi^\beta}{d\tau} = \frac{1}{m_\beta} P_\varphi^\beta \quad (0 < \tau < T,\ \beta = \pm 1), \tag{3.1}$$

$$\frac{dP_\varphi^\beta}{d\tau} = -\beta e \nabla_x \varphi(X_\varphi^\beta, \tau) + \frac{\beta e}{m_\beta c}[P_\varphi^\beta, B(X_\varphi^\beta)] \quad (0 < \tau < T,\ \beta = \pm 1) \tag{3.2}$$

with initial conditions

$$X_\varphi^\beta|_{\tau=0} = x \quad (\beta = \pm 1), \tag{3.3}$$

$$P_\varphi^\beta|_{\tau=0} = p \quad (\beta = \pm 1), \tag{3.4}$$

where $x \in \mathbb{R}^3_+$, $p \in \mathbb{R}^3$.

Since $\varphi \in C([0,T], C^{2+\sigma}(\overline{\mathbb{R}^3_+}))$, from the theorem on non-continuable solutions it follows that for any $x \in \mathbb{R}^3_+$ and $p \in \mathbb{R}^3$ there exists a unique non-continuable solution of (3.1)–(3.4) on some half-open interval $[0, T_\varphi^\beta(x,p))$ with $T_\varphi^\beta(x,p) \le T$. We denote this solution $(X_\varphi^\beta(x,p,\tau), P_\varphi^\beta(x,p,\tau))$.

Lemma 3.1. *Let* $\varphi \in M_{2+\sigma,R}$. *Then*

$$|P_\varphi^\beta(x,p,t)| < \rho + 3^{\frac{1}{2}} e R \quad (\beta = \pm 1) \tag{3.5}$$

for all $x \in \mathbb{R}^3_+$, $0 < t < T_\varphi^\beta(x,p)$, *and* $|p| < \rho$.

Proof. Multiply Eq. (3.2) by P_φ^β. Then we have

$$\frac{1}{2}\frac{d}{d\tau}|P_\varphi^\beta(x,p,\tau)|^2 = -\beta e(\nabla_x \varphi(X_\varphi^\beta, \tau), P_\varphi^\beta(x,p,\tau)) \le e|\nabla_x \varphi(X_\varphi^\beta, \tau)||P_\varphi^\beta(x,p,\tau)|.$$

From this inequality we obtain

$$\frac{d}{d\tau}|P_\varphi^\beta(x,p,\tau)| \le e|\nabla_x \varphi(X_\varphi^\beta, \tau)|. \tag{3.6}$$

Integrating (3.6) from 0 to t, we get

$$|P_\varphi^\beta(x,p,t)| \le |p| + e\int_0^t |\nabla_x \varphi(X_\varphi^\beta, \tau)| d\tau \le \rho + 3^{1/2} e \int_0^t \|\varphi(\cdot,\tau)\|_1 d\tau. \tag{3.7}$$

Since $\varphi \in M_{2+\sigma,R}$, inequality (3.7) implies (3.5).

We introduce the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

A multiplication by the matrix $R(\theta)$ corresponds to rotation by the angle θ in a plane. The following statement allows to apply properties of this operator to the investigation of trajectories of charged particles in the presence of the Lorentz force in (3.2).

Lemma 3.2. *The matrix* $R(\theta)$ *has the following properties*:

(a) $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2), \quad \theta_1, \theta_2 \in \mathbb{R}$;

(b) $R(\theta)^m = R(m\theta), \quad \theta \in \mathbb{R}, \quad m \in \mathbb{Z}$;

(c) $\dfrac{d}{d\theta} R(\theta) = R(\pi/2)R(\theta) = R(\theta + \pi/2), \quad \theta \in \mathbb{R}$;

(d) $|R(\theta)x| = |x|, \quad \theta \in \mathbb{R}, \quad x \in \mathbb{R}^2$;

(e) $\exp(tR(\theta)) = \exp(t\cos\theta)R(t\sin\theta)$.

Proof. Properties (a)–(d) are evident.

Let us prove property (e). Clearly

$$\exp(tR(\theta)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} R(\theta)^n \sum_{n=0}^{\infty} \frac{t^n}{n!} R(n\theta) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}. \tag{3.8}$$

Using the Euler formula, we obtain

$$\exp(t\exp(i\theta)) = \exp(t(\cos\theta + i\sin\theta)) = \exp(t\cos\theta)[\cos(t\sin\theta) + i\sin(t\sin\theta)]. \tag{3.9}$$

On the other hand, we have

$$\exp(t\exp(i\theta)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \exp(in\theta) \sum_{n=0}^{\infty} \frac{t^n}{n!} \cos(n\theta) + i \sum_{n=0}^{\infty} \frac{t^n}{n!} \sin(n\theta). \tag{3.10}$$

From (3.9) and (3.10) it follows that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} \cos(n\theta) = \exp(t\cos\theta)\cos(t\sin\theta), \\ &\sum_{n=0}^{\infty} \frac{t^n}{n!} \sin(n\theta) = \exp(t\cos\theta)\sin(t\sin\theta). \end{aligned} \tag{3.11}$$

Finally, (3.8) and (3.11) imply that

$$\exp(tR(\theta)) = \exp(t\cos\theta) \begin{pmatrix} \cos(t\sin\theta) & -\sin(t\sin\theta) \\ \sin(t\sin\theta) & \cos(t\sin\theta) \end{pmatrix} \exp(t\cos\theta)R(t\sin\theta).$$

Lemma 3.3. *Let Condition* 2.1 *hold. Then for all* $\varphi \in M_{2+\sigma,R}$ *a solution* $(X_\varphi^\beta(x,p,\tau), P_\varphi^\beta(x,p,\tau))$ *of problem* (3.1)–(3.4) *on the interval* $[0, T_\varphi^\beta(x,p))$ $(T_\varphi^\beta(x,p) \le T)$ *has the following properties: if* $5\delta/8 \le x_1 \le \varkappa - \delta/8$, $p \in B_\rho$, *then* $T_\varphi^\beta(x,p) = T$, $|X_{\varphi 1}^\beta(x,p,\tau) - x_1| < \delta/8$, *and* $P_\varphi^\beta(x,p,\tau) \in B_{\rho_1}$ *for all* $\tau \in [0,T)$.

Proof. 1. We prove that $T_\varphi^\beta(x,p) = T$ and $|X_{\varphi 1}^\beta(x,p,\tau) - x_1| < \delta/8$ for all $\tau \in [0, T_\varphi^\beta(x,p))$.

Assume to the contrary that either $T_\varphi^\beta(x,p) < T$ or $|X_{\varphi 1}^\beta(x,p,\tau_0) - x_1| \ge \delta/8$ for some $\tau_0 \in [0, T_\varphi^\beta(x,p))$.

Note that $T_\varphi^\beta(x,p) < T$ implies $\lim\limits_{\tau \to T_\varphi^\beta(x,p)-0} X_{\varphi 1}^\beta(x,p,\tau) = 0$ and hence $|X_{\varphi 1}^\beta(x,p,\tau_0) - x_1| \ge \delta/8$ for some $\tau_0 \in [0, T_\varphi^\beta)$. Since $X_{\varphi 1}^\beta(x,p,0) = x_1$, then for some τ_1, $0 < \tau_1 < T_\varphi^\beta(x,p)$, we have

$$|X_{\varphi 1}^\beta(x,p,\tau_1) - x_1| = \delta/8, \quad |X_{\varphi 1}^\beta(x,p,\tau) - x_1| < \delta/8 \quad (\tau \in [0,\tau_1)). \tag{3.12}$$

By virtue of Condition 2.1, we can rewrite equation (3.2) in the form

$$\frac{dP_{\varphi}^{\beta}(\tau)}{d\tau} = -\beta e \nabla_x \varphi(X_{\varphi}^{\beta}, \tau) + \frac{\beta e}{m^{\beta} c} \begin{pmatrix} 0 & h & 0 \\ -h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P_{\varphi}^{\beta}(\tau) \quad (\tau \in (0, \tau_1)).$$

Hence

$$\frac{d}{d\tau} \begin{pmatrix} P_{\varphi 1}^{\beta}(\tau) \\ P_{\varphi 2}^{\beta}(\tau) \end{pmatrix} + \frac{\beta e h}{m^{\beta} c} R(\frac{\pi}{2}) \begin{pmatrix} P_{\varphi 1}^{\beta}(\tau) \\ P_{\varphi 2}^{\beta}(\tau) \end{pmatrix} = -\beta e \nabla_{(x_1, x_2)} \varphi(X_{\varphi}^{\beta}, \tau) \quad (\tau \in (0, \tau_1)).$$

Multiplying the last equation by $\exp\left(\tau \frac{\beta e h}{m^{\beta} c} R\left(\frac{\pi}{2}\right)\right)$, we obtain

$$\frac{d}{d\tau}\left[\exp\left(\tau \frac{\beta e h}{m^{\beta} c} R\left(\frac{\pi}{2}\right)\right) \begin{pmatrix} P_{\varphi 1}^{\beta}(\tau) \\ P_{\varphi 2}^{\beta}(\tau) \end{pmatrix}\right] = -\beta e \exp\left(\tau \frac{\beta e h}{m^{\beta} c} R\left(\frac{\pi}{2}\right)\right) \nabla_{(x_1, x_2)} \varphi(X_{\varphi 1}^{\beta}, \tau) \quad (\tau \in (0, \tau_1)). \tag{3.13}$$

Using Lemma 3.2(e) and integrating (3.13) from 0 to t, $t \in (0, \tau_1)$, we have

$$\begin{pmatrix} P_{\varphi 1}^{\beta}(t) \\ P_{\varphi 2}^{\beta}(t) \end{pmatrix} = R\left(-t \frac{\beta e h}{m^{\beta} c}\right) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \beta e \int_0^t R\left((\tau - t) \frac{\beta e h}{m^{\beta} c}\right) \nabla_{(x_1, x_2)} \varphi(X_{\varphi}^{\beta}, \tau) d\tau.$$

Therefore from (3.1) we get

$$\begin{pmatrix} X_{\varphi 1}^{\beta}(\tau_1) \\ X_{\varphi 2}^{\beta}(\tau_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + I_1 + I_2, \tag{3.14}$$

where

$$I_1 = \frac{1}{m^{\beta}} \int_0^{\tau_1} R\left(-t \frac{\beta e h}{m^{\beta} c}\right) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} dt,$$

$$I_2 = -\frac{\beta e}{m^{\beta}} \int_0^{\tau_1} \int_0^t R\left((\tau - t) \frac{\beta e h}{m^{\beta} c}\right) \nabla_{(x_1, x_2)} \varphi(X_{\varphi}^{\beta}, \tau) d\tau dt.$$

We calculate I_1 and I_2.

By virtue of Lemma 3.2 (c), (a) we have

$$I_1 = \frac{c}{\beta e h}\left[-R\left(-t \frac{\beta e h}{m^{\beta} c} - \frac{\pi}{2}\right)\right]\Bigg|_{t=0}^{t=\tau_1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{c}{\beta e h}\left\{-R\left(-\tau_1 \frac{\beta e h}{m^{\beta} c}\right) + R(0)\right\} R\left(-\frac{\pi}{2}\right) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$= \frac{c}{\beta e h} \begin{pmatrix} 1 - \cos\left(\tau_1 \frac{e h}{m^{\beta} c}\right) & -\beta \sin\left(\tau_1 \frac{e h}{m^{\beta} c}\right) \\ \beta \sin\left(\tau_1 \frac{e h}{m^{\beta} c}\right) & 1 - \cos\left(\tau_1 \frac{e h}{m^{\beta} c}\right) \end{pmatrix} R\left(-\frac{\pi}{2}\right) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} := \begin{pmatrix} I_{11} \\ I_{12} \end{pmatrix}.$$

Hence we obtain

$$|I_{11}| \le \frac{c}{e h}\left(\left(1 - \cos\left(\tau_1 \frac{e h}{m^{\beta} c}\right)\right)^2 + \sin^2\left(\tau_1 \frac{e h}{m^{\beta} c}\right)\right)^{\frac{1}{2}} |p| \le \frac{2^{\frac{1}{2}} c}{e h}\left(1 - \cos\left(\tau_1 \frac{e h}{m^{\beta} c}\right)\right)^{\frac{1}{2}} |p|$$
$$\le \frac{2c}{e h} \psi(\tau_1)|p| \frac{2c}{e h} \psi(T)|p|. \tag{3.15}$$

On the other hand, using again Lemma 3.2(c), we see that

$$I_2 = -\frac{\beta e}{m^\beta}\int_0^{\tau_1}\left\{\int_\tau^{\tau_1} R\left((\tau - t)\frac{\beta eh}{m^\beta c}\right)dt\right\}\nabla_{(x_1,x_2)}\varphi(X_\varphi^\beta,\tau)d\tau$$

$$= \frac{c}{h}\int_0^{\tau_1}\left\{R\left((\tau-\tau_1)\frac{\beta eh}{m^\beta c}\right) - R(0)\right\}R\left(-\frac{\pi}{2}\right)\nabla_{(x_1,x_2)}\varphi(X_\varphi^\beta,\tau)d\tau := \begin{pmatrix} I_{21} \\ I_{22}\end{pmatrix}.$$

Then similarly to (3.15) we have

$$\begin{aligned}|I_{21}| &\le \frac{c}{h}\int_0^{\tau_1}\left(\left(1-\cos\left((\tau-\tau_1)\frac{eh}{m^\beta c}\right)\right)^2 + \sin^2\left((\tau-\tau_1)\frac{eh}{m^\beta c}\right)\right)^{\frac{1}{2}}|\nabla_{(x_1,x_2)}\varphi(X_\varphi^\beta,\tau)|d\tau \\ &\le \frac{2c}{h}\psi(T)\int_0^{\tau_1}|\nabla_{(x_1,x_2)}\varphi(X_\varphi^\beta,\tau)|d\tau \le \frac{2^{\frac{3}{2}}c}{h}\psi(T)\int_0^T\|\varphi(\cdot,\tau)\|_1 d\tau.\end{aligned} \tag{3.16}$$

From (3.12), (3.14)–(3.16) it follows that

$$\begin{aligned}\frac{\delta}{8} = |X_{\varphi 1}^\beta(x,p,\tau_1) - x_1| &\le |I_{11}| + |I_{21}| \le \psi(T)\frac{2c}{eh}\left(\rho + 2^{\frac{1}{2}}e\int_0^T\|\varphi(\cdot,\tau)\|_1 d\tau\right) \\ &\le \psi(T)\frac{2c}{eh}\left(\rho + 2^{\frac{1}{2}}eR\right).\end{aligned} \tag{3.17}$$

On the other hand, (2.8) imply that

$$\frac{2c}{eh}\psi(T)\left(\rho + 3^{\frac{1}{2}}eR\right) < \frac{\delta}{8}.$$

This contradicts (3.17). Thus we have proved that $T_\varphi^\beta(x,p) = T$ and $|X_{\varphi 1}^\beta(x,p,\tau) - x_1| < \delta/8$ for all $\tau \in [0,T)$.

2. From (2.9), (2.10) it follows that $\rho + 3^{1/2}eR < \rho_1$. Therefore, by virtue of Lemma 3.1, $|P_\varphi^\beta(x,p,t)| < \rho_1$ for all $x_1 \in \left[\frac{5\delta}{8}, \varkappa - \frac{\delta}{8}\right]$, $p \in B_\rho$, and $t \in [0,T)$.

Corollary 3.1. Assume that the following condition holds.

Condition 3.1. Let $B \in \hat{C}^{1+\sigma}(\overline{\mathbb{R}_+^3})$, and let $B(x) = (0,0,h)$ for $\delta/4 \le x_1 \le \varkappa$, where

$$\frac{16c}{e\delta}(\rho + \sqrt{3}eR) < h, \tag{3.18}$$

$\varkappa, \delta, \rho, R, h > 0$ do not depend on x.

Then, for any $\varphi \in M_{2+\sigma,R}$, $x \in \mathbb{R}_{7\delta/8}^3$, and $p \in B_\rho$, we have $T_\varphi^\beta(x,p) = \infty$, and $X_\varphi^\beta(x,p,\tau) \in \mathbb{R}_{3\delta/4}^3$, $P_\varphi^\beta(x,p,\tau) \in B_{\rho_1}$ for all $\tau \in [0,\infty)$.

3.2. We now consider system of differential equations (3.1), (3.2) on the interval $(0,t)$, $0 < t \le T$, with initial conditions

$$X_\varphi^\beta|_{\tau=t} = y \quad (\beta = \pm 1), \tag{3.19}$$

$$P_\varphi^\beta|_{\tau=t} = q \quad (\beta = \pm 1). \tag{3.20}$$

By virtue of the theorem on non-continuable solutions, it follows that for any $y \in \mathbb{R}_+^3$ and $q \in \mathbb{R}^3$ there is a unique non-continuable solution of problem (3.1), (3.2), (3.19), (3.20) for $\tau \in (T_\varphi^\beta(y,q,t),t]$, $0 \le T_\varphi^\beta(y,q,t) < t$. Denote this solution by $(X_\varphi^\beta(y,q,t,\tau)\ P_\varphi^\beta(y,q,t,\tau))$.

Similarly to Lemma 3.3 we can prove the following statement.

Lemma 3.4. *Let Condition* 2.1 *hold. Then for any* $\varphi \in M_{2+\sigma,R}$ *a solution* $(X_{\varphi}^{\beta}(y,q,t,\tau),\ P_{\varphi}^{\beta}(y,q,t,\tau))$ *of problem* (3.1), (3.2), (3.19), (3.20) *on the interval* $(T_{\varphi}^{\beta}(y,q,t),t]$ $(0 \le T_{\varphi}^{\beta}(y,q,t))$ *has the following properties*: *if* $5\delta/8 \le y_1 \le \varkappa - \delta/8$, $q \in B_{\rho_1}$, *and* $0 < t \le T$, *then* $T_{\varphi}^{\beta}(y,q,t) = 0$, $|X_{\varphi 1}^{\beta}(y,q,t,\tau) - y_1| < \delta/8$ *and* $P_{\varphi}^{\beta}(y,q,t,\tau) \in B_{\rho_2}$ *for all* $\tau \in (0,t]$.

Corollary 3.2. Let Condition 3.1 hold. Then for any $\varphi \in M_{2+\sigma,R}$, $y \in \mathbb{R}^3_{7\delta/8}$, $q \in B_{\rho_1}$, and $0 < t \le T$, we have $T_{\varphi}^{\beta}(y,q,t) = 0$, $X_{\varphi}^{\beta}(y,q,t,\tau) \in \mathbb{R}^3_{3\delta/4}$, and $P_{\varphi}^{\beta}(y,q,t,\tau) \in B_{\rho_2}$.

4. CHARGED PARTICLES DENSITY DISTRIBUTIONS

4.1. Denote

$$\Omega_0^{\beta} = (\mathbb{R}^3_{3\delta/4} \cap B_{\varkappa^{\beta}}) \times B_{\rho_1}, \tag{4.1}$$

where $\varkappa^{\beta} = \varkappa + T\rho_1/m^{\beta}$.

Let Condition 2.1 hold, and let $\varphi \in M_{2+\sigma,R}$. Then from Lemma 3.4 it follows that for any $(y,q) \in \Omega_0^{\beta}$ and $0 < t \le T$, there is a unique non-continuable solution $(X_{\varphi}^{\beta}(y,q,t,\tau),\ P_{\varphi}^{\beta}(y,q,t,\tau))$ of problem (3.1), (3.2), (3.19), (3.20) on the half-open interval $(0,t]$. Extending the functions $X_{\varphi}^{\beta}(y,q,t,\tau)$, $P_{\varphi}^{\beta}(y,q,t,\tau)$ by continuity at $\tau = 0$, we set $\hat{X}_{\varphi}^{\beta}(y,q,t) = X_{\varphi}^{\beta}(y,q,t,0)$ and $\hat{P}_{\varphi}^{\beta}(y,q,t) = P_{\varphi}^{\beta}(y,q,t,0)$.

Given $0 < t \le T$, consider the map $\hat{S}_{\varphi,\tau}^{\beta}(y,q) : \Omega_0^{\beta} \to \Omega_{\varphi,t}^{\beta} = \{(x,p) : (x,p) = \hat{S}_{\varphi,t}(y,q),\ (y,q) \in \Omega_0^{\beta}\}$ given by

$$\hat{S}_{\varphi,t}^{\beta}(y,q) = (\hat{X}_{\varphi}^{\beta}(y,q,t), \hat{P}_{\varphi}^{\beta}(y,q,t)).$$

Since $\varphi \in M_{2+\sigma,R}$ and $B \in \hat{C}^{1+\sigma}(\overline{\mathbb{R}^3_+})$, by theorem on differentiability of the solutions with respect to the initial data, the function $\hat{S}_{\varphi,t}^{\beta}(y,q)$ is continuously differentiable with respect to y and q on the set Ω_0^{β}.

Let $\hat{S}_{\varphi,0}^{\beta}(x,p) = (x,p)$.

Clearly, for any $0 \le t < T$, the map $S_{\varphi,t}^{\beta}$: $\Omega_{\varphi,t}^{\beta} \to \Omega_0^{\beta}$ given by

$$S_{\varphi,t}^{\beta}(x,p) = (X_{\varphi}^{\beta}(x,p,t), P_{\varphi}^{\beta}(x,p,t))$$

is the inverse map to $\hat{S}_{\varphi,t}^{\beta}$, i.e.,

$$\hat{S}_{\varphi,t}^{\beta}(S_{\varphi,t}^{\beta}(x,p)) = (x,p) \qquad ((x,p) \in \Omega_{\varphi,t}^{\beta}). \tag{4.2}$$

We extend the map $S_{\varphi,t}^{\beta}$ by continuity at $t = T$.

Since $\varphi \in M_{2+\sigma,R}$ and $B \in \hat{C}^{1+\sigma}(\overline{\mathbb{R}^3_+})$, by theorem on differentiability of solutions with respect to initial data, the function $S_{\varphi,t}(x,p)$ is continuously differentiable with respect to x, p, and t for any $t \in [0,T]$ and $(x,p) \in \Omega_{\varphi,t}^{\beta}$. Therefore from (4.2) and continuous differentiability of $\hat{S}_{\varphi,t}^{\beta}$ with respect to y and q it follows that the function $\hat{S}_{\varphi,t}^{\beta}$, $y,q \in \Omega_0^{\beta}$, is continuously differentiable with respect to t.

Lemma 4.1. *Let Condition* 2.1 *hold, and let* $\varphi \in M_{2+\sigma,R}$. *Then*

$$\left|\frac{\partial \hat{X}_{\varphi}^{\beta}(x,p,t)}{\partial x_j}\right| + \left|\frac{\partial \hat{P}_{\varphi}^{\beta}(x,p,t)}{\partial x_j}\right| c_0 \qquad ((x,p) \in \Omega_0^{\beta},\ 0 < t < T,\ j = 1,2,3), \tag{4.3}$$

where

$$c_0 = \exp\left(\max\left\{3eR + \frac{\sqrt{3}e\rho_2}{cm^{-1}}\langle B\rangle_1 T, \frac{T}{m^{-1}}\right\}\right). \tag{4.4}$$

Proof. Let $(x, p) \in \Omega_0^\beta$. The variational equations for system (3.1), (3.2) have the form

$$\frac{d}{d\tau}\left(\frac{\partial X_\varphi^\beta}{\partial x_j}\right) = \frac{1}{m^\beta}\frac{\partial P_\varphi^\beta}{\partial x_j} \quad (0 < \tau < t), \tag{4.5}$$

$$\frac{d}{d\tau}\left(\frac{\partial P_\varphi^\beta}{\partial x_j}\right) = -\beta e \sum_{k=1}^{3} \nabla_{X_\varphi^\beta}\left(\frac{\partial \varphi(X_\varphi^\beta, \tau)}{\partial X_{\varphi k}^\beta}\right)\frac{\partial X_{\varphi k}^\beta}{\partial x_j}$$
$$+\frac{\beta e}{m^\beta c}\left\{\left[\frac{\partial P_\varphi^\beta}{\partial x_j}, B(X_\varphi^\beta)\right] + \left[P_\varphi^\beta, \sum_{k=1}^{3}\frac{\partial B(X_\varphi^\beta)}{\partial X_{\varphi k}^\beta}\frac{\partial X_{\varphi k}^\beta}{\partial x_j}\right]\right\} \quad (0 < \tau < t). \tag{4.6}$$

By virtue of (3.19), (3.20) with $y = x$ and $q = p$, the initial conditions for system (4.4), (4.5) will take the form

$$\left.\frac{\partial X_{\varphi i}^\beta}{\partial x_j}\right|_{\tau=t} = \delta_{ij}, \quad \frac{\partial P_{\varphi i}^\beta}{\partial x_j}\Big|_{\tau=t} = 0 \quad (i = 1, 2, 3). \tag{4.7}$$

Multiplying both sides of (4.5) and (4.6) by $\frac{\partial X_\varphi^\beta}{\partial x_j}$ and $\frac{\partial P_\varphi^\beta}{\partial x_j}$, respectively, we pass to the new variable $\xi = \tau$ and integrate the obtained equalities with respect to ξ from τ to t, taking into account initial conditions (4.7). Then we change to the new variable $s = t - \xi$. Let $\tilde{X}_\varphi^\beta(s) = X_\varphi^\beta(t-s)$ and $\tilde{P}_\varphi^\beta(s) = P_\varphi^\beta(t-s)$, and let $\tau_1 = t - \tau$. Then we have

$$\left|\frac{\partial \tilde{X}_\varphi^\beta(\tau_1)}{\partial x_j}\right| \le 1 + \frac{1}{m^\beta}\int_0^{\tau_1}\left|\frac{\partial \tilde{P}_\varphi^\beta(s)}{\partial x_j}\right| ds \quad (0 < \tau_1 < t),$$

$$\left|\frac{\partial \tilde{P}_\varphi^\beta(\tau_1)}{\partial x_j}\right| \le 3e\int_0^{\tau_1}\|\varphi(\cdot, t-s)\|_2\left|\frac{\partial \tilde{X}_\varphi^\beta(s)}{\partial x_j}\right| ds + \frac{\sqrt{3}e\rho_2}{m^\beta c}\langle B\rangle_1\int_0^{\tau_1}\left|\frac{\partial \tilde{X}_\varphi^\beta(s)}{\partial x_j}\right| ds \quad (0 < \tau_1 < t).$$

From these inequalities and the Gronwall lemma we obtain

$$\left|\frac{\partial X_\varphi^\beta(\tau)}{\partial x_j}\right| + \left|\frac{\partial P_\varphi^\beta(\tau)}{\partial x_j}\right| = \left|\frac{\partial \tilde{X}_\varphi^\beta(\tau_1)}{\partial x_j}\right| + \left|\frac{\partial \tilde{P}_\varphi^\beta(\tau_1)}{\partial x_j}\right| \le c_0 \quad (0 \le \tau < t). \tag{4.8}$$

Setting $\tau = 0$ in the functions

$$\left|\frac{\partial X_\varphi^\beta(\tau)}{\partial x_j}\right| = \left|\frac{\partial X_\varphi^\beta(x, p, t, \tau)}{\partial x_j}\right| \quad \text{and} \quad \left|\frac{\partial P_\varphi^\beta(\tau)}{\partial x_j}\right| = \left|\frac{\partial P_\varphi^\beta(x, p, t, \tau)}{\partial x_j}\right|,$$

we obtain (4.3) from (4.8).

4.2. Denote

$$\mathcal{D}_0^\beta = (\mathbb{R}^3_{7\delta/8} \cap B_{\varkappa^\beta - \delta/8}) \times B_{\rho_1}, \tag{4.9}$$

where $\varkappa^\beta > 0$ is given by (4.1).

Lemma 4.2. *Let Conditions* 2.1 *and* 2.2 *hold. Then, for any* $\varphi \in M_{2+\sigma,R}$ *and* $0 < t \le T$, *we have* $\operatorname{supp} f_0^\beta(\hat{S}_{\varphi,t}^\beta(x, p)) \subset \mathcal{D}_0^\beta$.

Proof. It is sufficient to show that $S^{\beta}_{\varphi,t}(\operatorname{supp} f_0^{\beta}) \subset \mathcal{D}_0^{\beta}$. By virtue of Condition 2.2 $\operatorname{supp} f_0^{\beta} \subset \mathcal{D}_0$. We prove that $S^{\beta}_{\varphi,t}(\mathcal{D}_0) \subset \mathcal{D}_0^{\beta}$. Let $(x, p) \in \mathcal{D}_0$. Then from Lemma 3.3 it follows that $S^{\beta}_{\varphi,t}(x, p) \in \mathbb{R}^3_{7\delta/8} \times B_{\rho_1}$. By assumption $|x| < \varkappa - \delta/8$. Therefore, since $|P^{\beta}_{\varphi}(x, p, \tau)| \le \rho_1$ $(0 \le \tau \le t)$, from (3.1) we obtain

$$|X^{\beta}_{\varphi}(x, p, t)| \le |x| + \frac{1}{m^{\beta}} \int_0^t |P^{\beta}_{\varphi}(x, p, \tau)| d\tau < \varkappa - \frac{\delta}{8} + \frac{T\rho_1}{m^{\beta}} = \varkappa^{\beta} - \frac{\delta}{8}.$$

Let Conditions 2.1 and 2.2 hold.

We define the function $f^{\beta}_{\varphi}(x, p, t)$ by the formula

$$f^{\beta}_{\varphi}(x, p, t) = \begin{cases} f_0^{\beta}(\hat{S}^{\beta}_{\varphi,t}(x, p)) & ((x, p) \in \mathcal{D}_0^{\beta},\ 0 \le t \le T), \\ 0 & ((x, p) \in (\mathbb{R}^3_+ \times \mathbb{R}^3) \backslash \mathcal{D}_0^{\beta},\ 0 \le t \le T), \end{cases} \tag{4.10}$$

where $\varphi \in M_{2+\sigma,R}$.

By virtue of Lemma 4.2, $\operatorname{supp} f_0^{\beta}(\hat{S}^{\beta}_{\varphi,t}(x, p)) \subset \mathcal{D}_0^{\beta}$. Hence, using the characteristics method and continuous differentiability of the functions $\hat{S}^{\beta}_{\varphi,t}(x, p)$ with respect to x, p, t we see that for any given function $\varphi \in M_{2+\sigma,R}$ there exists a unique solution of problem (1.2), (1.3) in $C^1(\overline{\mathbb{R}^3_+} \times \mathbb{R} \times [0, T])$. This solution is given by (4.10).

Denote

$$F_{\varphi}(x, t) = \int_{\mathbb{R}^3} \sum_{\beta = \pm 1} \beta f^{\beta}_{\varphi}(x, p, t) dp \quad (x \in \overline{\mathbb{R}^3_+},\ 0 \le t \le T). \tag{4.11}$$

Remark 4.1. By virtue of Lemma 4.2 and formula (4.9), $\operatorname{supp} f_0^{\beta}(\hat{S}^{\beta}_{\varphi,t}(x, p)) \subset (\mathbb{R}^3_{7\delta/8} \cap B_{\varkappa^{\beta} - \delta/8}) \times B_{\rho_1}$. Hence in (4.11) we integrate over B_{ρ_1}. Therefore the integral in (4.11) exists.

5. HÖLDER ESTIMATES

5.1. In this section we prove Hölder estimates for functions $F_{\varphi}(x, t)$ $(\varphi \in M_{2+\sigma,R})$.

Repeating the proof of Lemma 4.2 in [31], we obtain the following statement.

Lemma 5.1. *Let Conditions* 2.1 *and* 2.2 *hold, and let* $\varphi \in M_{2+\sigma,R}$. *Then*

$$F_{\varphi} \in C([0, T], C^{\sigma}(\overline{\mathbb{R}^3_+})).$$

Denote $m_s = \max_{\beta} \|f_0^{\beta}\|_s$ $(s \ge 0)$.

Lemma 5.2. *Let Conditions* 2.1 *and* 2.2 *hold. Then, for any* $\varphi \in M_{2+\delta,R}$, *we have*

$$\|F_{\varphi}\|_{L_1((0,T), C^{\sigma}(\overline{\mathbb{R}^3_+}))} \le c_1 m_{\sigma}, \tag{5.1}$$

where $c_1 = 2|B_{\rho_1}|(1 + 3^{\sigma/2} c_0^{\sigma})T$, $c_0 > 0$ *is the constant from Lemma* 4.1.

Proof. From formulas (4.10), (4.11), and Remark 4.1 it follows that

$$|F_{\varphi}(x, t)| \le \sum_{\beta} \int_{|p| < \rho_1} |f^{\beta}_{\varphi}(x, p, t)| dp \le 2|B_{\rho_1}| m_0 \quad (x \in \overline{\mathbb{R}^3_+},\ 0 \le t \le T). \tag{5.2}$$

On the other hand, using the Taylor formula and Lemma 4.1, we obtain

$$\begin{aligned} |\delta_{\Delta x} F(x, t)| &\le \sum_{\beta} \int_{|p| < \rho_1} |\delta_{\Delta x} f^{\beta}_{\varphi}(x, p, t)| dp \le m_{\sigma} \sum_{\beta} \int_{|p| < \rho_1} |\delta_{\Delta x} \hat{S}^{\beta}_{\varphi,t}(x, p)|^{\sigma} dp \\ &\le 2 \cdot 3^{\sigma/2} |B_{\rho_1}| c_0^{\sigma} m_{\sigma} |\Delta x|^{\sigma} \quad (x \in \overline{\mathbb{R}^3_+},\ 0 \le t \le T), \end{aligned} \tag{5.3}$$

where $c_0 > 0$ is the constant from Lemma 4.1. Therefore we obtain

$$\|F_{\varphi}(\cdot,t)\|_{\sigma} \le 2|B_{\rho_1}|m_{\sigma}(1+3^{\sigma/2}c_0^{\sigma}). \tag{5.4}$$

Integrating (5.4), we obtain (5.1).

Lemma 5.3. *Let Conditions* 2.1 *and* 2.2 *hold. Then, for any* $\varphi_1, \varphi_2 \in M_{2+\sigma,R}$ *and* $0 \le t \le T$*, we have*

$$\|F_{\varphi_1}(\cdot,t) - F_{\varphi_2}(\cdot,t)\|_{\sigma} \le c_2 m_{1+\sigma}\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^2(\overline{\mathbb{R}^3_+}))}, \tag{5.5}$$

where $c_2 = c_2(T,\delta,\rho,R,\|B\|_{1+\sigma},\sigma) > 0$ *does not depend on* φ_1 *and* φ_2.

Proof. 1. From Remark 4.1 and the Taylor formula we derive the following estimate

$$\begin{aligned}|F_{\varphi_1}(x,t) - F_{\varphi_2}(x,t)| &\le \sum_{\beta}\int_{|p|<\rho_1}\left|f^{\beta}_{\varphi_1}(x,p,t) - f^{\beta}_{\varphi_2}(x,p,t)\right|dp \\ &\le \sqrt{3}m_1\sum_{\beta}\int_{|p|<\rho_1}\left\{\left|\hat{X}^{\beta}_{\varphi_1}(x,p,t) - \hat{X}^{\beta}_{\varphi_2}(x,p,t)\right| + \left|\hat{P}^{\beta}_{\varphi_1}(x,p,t) - \hat{P}^{\beta}_{\varphi_2}(x,p,t)\right|\right\}dp \quad (0 \le t \le T).\end{aligned} \tag{5.6}$$

We now estimate the right-hand side of (5.6). Let

$$\{X^{\beta}_{\varphi_l}(\tau), P^{\beta}_{\varphi_l}(\tau)\} = \{X^{\beta}_{\varphi_l}(x,p,t,\tau), P^{\beta}_{\varphi_l}(x,p,t,\tau)\},$$

where $(x,p) \in \mathcal{D}^{\beta}_0$, $\tau \in (0,t]$, be the solution of system (3.1), (3.2) for $\varphi = \varphi_l$, $l = 1,2$, with initial conditions

$$X^{\beta}_{\varphi_l}(x,p,t,\tau)|_{\tau=t} = x, \quad P^{\beta}_{\varphi_l}(x,p,t,\tau)|_{\tau=t} = p. \tag{5.7}$$

Then, using again the Taylor formula, we have

$$\frac{d}{d\tau}(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2}) = \frac{1}{m^{\beta}}(P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}) \quad (0 < \tau < t), \tag{5.8}$$

$$\begin{aligned}\frac{d}{d\tau}(P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}) &= -\beta e\sum_{j=1}^{3}\int_0^1\left(\frac{\partial}{\partial X_j}\nabla_X\varphi_1\right)(X^{\beta}_{\varphi_2} + \theta(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2}),\tau)d\theta \\ &\times(X^{\beta}_{\varphi_1,j} - X^{\beta}_{\varphi_2,j}) - \beta e(\nabla_X\varphi_1(X^{\beta}_{\varphi_2},\tau) - \nabla_X\varphi_2(X^{\beta}_{\varphi_2},\tau)) \\ &+ \frac{\beta e}{m^{\beta}c}[P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}, B(X^{\beta}_{\varphi_1})] + \frac{\beta e}{m^{\beta}c}[P^{\beta}_{\varphi_2}, B(X^{\beta}_{\varphi_1}) - B(X^{\beta}_{\varphi_2})] \quad (0 < \tau < t).\end{aligned} \tag{5.9}$$

We multiply Eqs. (5.8) and (5.9) by $X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2}$ and $P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}$, respectively, pass to the new variable $s = \tau$, and integrate with respect to s from τ to t, $0 < \tau < t$. Then, taking into account initial conditions (5.7), by Lemma 3.4, we obtain

$$\begin{aligned}|X^{\beta}_{\varphi_1}(\tau) - X^{\beta}_{\varphi_2}(\tau)| &\le \frac{1}{m^{\beta}}\int_{\tau}^{t}|P^{\beta}_{\varphi_1}(s) - P^{\beta}_{\varphi_2}(s)|ds, \\ |P^{\beta}_{\varphi_1}(\tau) - P^{\beta}_{\varphi_2}(\tau)| &\le 3e\int_{\tau}^{t}\|\varphi_1(\cdot,s)\|_2|X^{\beta}_{\varphi_1}(s) - X^{\beta}_{\varphi_2}(s)|ds \\ &+ \sqrt{3}e\int_{\tau}^{t}\|(\varphi_1 - \varphi_2)(\cdot,s)\|_1 ds + \sqrt{3}\frac{e\rho_2}{cm^{\beta}}\langle B\rangle_1\int_{\tau}^{t}|X^{\beta}_{\varphi_1}(s) - X^{\beta}_{\varphi_2}(s)|ds.\end{aligned} \tag{5.10}$$

We introduce the new variables $s_1 = t - s$, $\tau_1 = t - \tau$ and define $\tilde{X}^{\beta}_{\varphi_j}(\tau_1) = X^{\beta}_{\varphi_j}(\tau)$, $\tilde{P}^{\beta}_{\varphi_j}(\tau_1) = P^{\beta}_{\varphi_j}(\tau)$, $j = 1, 2$. Then from inequalities (5.10), formula (2.10) for ρ_2 and ρ_1, and the Gronwall lemma we derive the following inequality:

$$|X^{\beta}_{\varphi_1}(\tau) - X^{\beta}_{\varphi_2}(\tau)| + |P^{\beta}_{\varphi_1}(\tau) - P^{\beta}_{\varphi_2}(\tau)| = |\tilde{X}^{\beta}_{\varphi_1}(\tau_1) - \tilde{X}^{\beta}_{\varphi_2}(\tau_1)| + |\tilde{P}^{\beta}_{\varphi_1}(\tau_1) - \tilde{P}^{\beta}_{\varphi_2}(\tau_1)| \le k_1\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^1(\overline{\mathbb{R}^3_+}))}, \tag{5.11}$$

where $k_1 = k_1(T, \delta, \rho, R, \|B\|_1) > 0$ does not depend on φ_1, φ_2.

Putting $\tau = 0$ in (5.11), we have

$$|X^{\beta}_{\varphi_1}(0) - X^{\beta}_{\varphi_2}(0)| + |P^{\beta}_{\varphi_1}(0) - P^{\beta}_{\varphi_2}(0)| = |\hat{X}^{\beta}_{\varphi_1}(x, p, t) - \hat{X}^{\beta}_{\varphi_2}(x, p, t)| + |\hat{P}^{\beta}_{\varphi_1}(x, p, t) - \hat{P}^{\beta}_{\varphi_2}(x, p, t)| \le k_1\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^1(\overline{\mathbb{R}^3_+}))}. \tag{5.12}$$

From inequalities (5.6) and (5.12) it follows that

$$\sup_{x\in R^3_+}|F_{\varphi_1}(x,t) - F_{\varphi_2}(x,t)| \le 4m_1|B_{\rho_1}|k_1\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^1(\overline{\mathbb{R}^3_+}))}. \tag{5.13}$$

2. Let $(\delta_{\Delta x} f)(x) = f(x + \Delta x) - f(x)$, where $|\Delta x| < \delta/8$, and the function $f(\cdot)$ is defined at the points x, $x + \Delta x$. For any domain $\mathcal{D} \subset \mathbb{R}^3$ denote $\mathcal{D}^{\varepsilon} = \{x \in \mathbb{R}^3\colon \operatorname{dist}(x, \mathcal{D}) < \varepsilon\}$, $\varepsilon > 0$. From (4.1) and (4.9) we have

$$(R^3_{7\delta/8} \cap B_{\varkappa^{\beta}-\delta/8})^{\delta/8} \times B_{\rho_1} \subset \Omega^{\beta}_0. \tag{5.14}$$

Therefore, using Remark 4.1 and the Taylor formula, we obtain

$$|\delta_{\Delta x}(F_{\varphi_1}(x,t) - F_{\varphi_2}(x,t))| \le \sum_{\beta}\int_{|p|<\rho_1}|\delta_{\Delta x}(f^{\beta}_{\varphi_1}(x, p, t) - f^{\beta}_{\varphi_2}(x, p, t))|dp \le \sum_{\beta}\sum_{j=1,2} I^{\beta}_j, \tag{5.15}$$

where

$$I^{\beta}_1 = \int_{|p|<\rho_1} dp\int_0^1\sum_{j=1}^3\{|\delta_{\Delta x} f^{\beta}_{0X_j}(\hat{S}^{\beta}_{\varphi_2,t} + \theta(\hat{S}^{\beta}_{\varphi_1,t} - \hat{S}^{\beta}_{\varphi_2,t}))(\hat{X}^{\beta}_{\varphi_1,j}(x + \Delta x, p, t) - \hat{X}^{\beta}_{\varphi_2,j}(x + \Delta x, p, t))| + |\delta_{\Delta x} f^{\beta}_{0P_j}(\hat{S}_{\varphi_2,t} + \theta(\hat{S}^{\beta}_{\varphi_1,t} - \hat{S}^{\beta}_{\varphi_2,t})) \times (\hat{P}^{\beta}_{\varphi_1,j}(x + \Delta x, p, t) - \hat{P}^{\beta}_{\varphi_2,j}(x + \Delta x, p, t))|\}d\theta,$$

$$I^{\beta}_2 = \int_{|p|<\rho_1} dp\int_0^1\sum_{j=1}^3\{|f^{\beta}_{0X_j}(\hat{S}^{\beta}_{\varphi_2,t} + \theta(\hat{S}^{\beta}_{\varphi_1,t} - \hat{S}^{\beta}_{\varphi_2,t}))\delta_{\Delta x}(\hat{X}^{\beta}_{\varphi_1,j} - \hat{X}^{\beta}_{\varphi_2,j})| + |f^{\beta}_{0P_j}(\hat{S}^{\beta}_{\varphi_2,t} + \theta(\hat{S}^{\beta}_{\varphi_1,t} - \hat{S}^{\beta}_{\varphi_2,t}))\delta_{\Delta x}(\hat{P}^{\beta}_{\varphi_1,j} - \hat{P}^{\beta}_{\varphi_2,j})|\}d\theta.$$

Let us estimate I^{β}_1. By virtue of inequality (5.11) and Lemma 4.1, we have

$$I^{\beta}_1 \le k_2 m_{1+\sigma}|B_{\rho_1}|c^{\sigma}_0\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^1(\overline{\mathbb{R}^3_+}))}|\Delta x|^{\sigma}, \tag{5.16}$$

where $k_2 = k_2(T, \delta, \rho, R, \|B\|_1) > 0$ does not depend on ϕ_1 and ϕ_2.

Clearly,

$$I^{\beta}_2 \le 2m_1|B_{\rho_1}|\{|\delta_{\Delta x}(\hat{X}^{\beta}_{\varphi_1} - \hat{X}^{\beta}_{\varphi_2})| + |\delta_{\Delta x}(\hat{P}^{\beta}_{\varphi_1} - \hat{P}^{\beta}_{\varphi_2})|\}. \tag{5.17}$$

To estimate the right-hand side of (5.17), we apply the operator $\delta_{\Delta x}$ to both parts of system (5.8), (5.9). Then we have

$$\frac{d}{d\tau}\delta_{\Delta x}(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2}) = \frac{1}{m^{\beta}}\delta_{\Delta x}(P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}), \quad 0 < \tau < t,$$
$$\frac{d}{d\tau}\delta_{\Delta x}(P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}) = \beta\sum_{\mu=1}^{7} J^{\beta}_{\mu}, \quad 0 < \tau < t, \tag{5.18}$$

where

$$J^{\beta}_1 = -e\sum_{i=1}^{3}\int_0^1 (\delta_{\Delta x}\frac{\partial}{\partial X_j}\nabla_X\varphi_1)(X^{\beta}_{\varphi_2} + \theta(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2}),\tau)d\theta \times (X^{\beta}_{\varphi_1,j}(x+\Delta x, p, t, \tau) - X^{\beta}_{\varphi_2,j}(x+\Delta x, p, t, \tau)),$$

$$J^{\beta}_2 = -e\sum_{j=1}^{3}\int_0^1 (\frac{\partial}{\partial X_j}\nabla_X\varphi_1)(X^{\beta}_{\varphi_2} + \theta(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2}),\tau)d\theta\,\delta_{\Delta x}(X^{\beta}_{\varphi_1,j} - X^{\beta}_{\varphi_2,j}),$$

$$J^{\beta}_3 = -e\delta_{\Delta x}(\nabla_X\varphi_1(X^{\beta}_{\varphi_2},\tau) - \nabla_X\varphi_2(X^{\beta}_{\varphi_2},\tau)),$$

$$J^{\beta}_4 = \frac{e}{m^{\beta}c}[\delta_{\Delta x}(P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}), B(X^{\beta}_{\varphi_1}(x+\Delta x, p, t, \tau))],$$

$$J^{\beta}_5 = \frac{e}{m^{\beta}c}[P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2}, \delta_{\Delta x}B(X^{\beta}_{\varphi_1})],$$

$$J^{\beta}_6 = \frac{e}{m^{\beta}c}[\delta_{\Delta x}P^{\beta}_{\varphi_2}, B(X^{\beta}_{\varphi_1}(x+\Delta x, p, t, \tau)) - B(X^{\beta}_{\varphi_2}(x+\Delta x, p, t, \tau))],$$

$$J^{\beta}_7 = \frac{e}{m^{\beta}c}[P^{\beta}_{\varphi_2}, \delta_{\Delta x}(B(X^{\beta}_{\varphi_1}) - B(X^{\beta}_{\varphi_2}))].$$

From (5.11), (4.8) and Lemma 3.4 we obtain

$$|J^{\beta}_1| \le k_3\|\varphi_1(\cdot,\tau)\|_{2+\sigma}\|\varphi_1 - \varphi_2\|_{L_1((0,T),C^1(\overline{\mathbb{R}^3_+}))}|\Delta x|^{\sigma},$$
$$|J^{\beta}_2| \le k_3\|\varphi_1(\cdot,\tau)\|_2|\delta_{\Delta x}(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2})|,$$
$$|J^{\beta}_3| \le k_3\|(\varphi_1 - \varphi_2)(\cdot,\tau)\|_2|\Delta x|,$$
$$|J^{\beta}_4| \le k_3|\delta_{\Delta x}(P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2})|, \tag{5.19}$$
$$|J^{\beta}_5| \le k_3\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^1(\overline{\mathbb{R}^3_+}))}|\Delta x|,$$
$$|J^{\beta}_6| \le k_3\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^1(\overline{\mathbb{R}^3_+}))}|\Delta x|,$$
$$|J^{\beta}_7| \le k_3(\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^1(\overline{\mathbb{R}^3_+}))}|\Delta x|^{\sigma} + |\delta_{\Delta x}(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2})|),$$

where $k_3 = k_3(T,\delta,\rho,R,\|B\|_{1+\sigma}) > 0$ does not depend on φ_1 and φ_2.

By virtue of (5.7), the initial conditions for system (5.18) have the form

$$\delta_{\Delta x}(X^{\beta}_{\varphi_1} - X^{\beta}_{\varphi_2})|_{\tau=t} = 0, \quad \delta_{\Delta x}(P^{\beta}_{\varphi_1} - P^{\beta}_{\varphi_2})|_{\tau=t} = 0. \tag{5.20}$$

Integrating system (5.18) from τ to t, $0 < \tau < t$, with initial conditions (5.20), by virtue of (5.19), we have

$$|\delta_{\Delta x}(X^{\beta}_{\varphi_1}(\tau) - X^{\beta}_{\varphi_2}(\tau))| \le \frac{1}{m^{\beta}}\int_{\tau}^{t}|\delta_{\Delta x}(P^{\beta}_{\varphi_1}(\tau) - P^{\beta}_{\varphi_2}(\tau))|ds,$$
$$|\delta_{\Delta x}(P^{\beta}_{\varphi_1}(\tau) - P^{\beta}_{\varphi_2}(\tau))| \le k_4(\|\varphi_1 - \varphi_2\|_{L_1((0,t),C^2(\overline{\mathbb{R}^3_+}))}|\Delta x|^{\sigma} \tag{5.21}$$
$$+ \int_{\tau}^{t}(|\delta_{\Delta x}(X^{\beta}_{\varphi_1}(s) - X^{\beta}_{\varphi_2}(s))|ds + |\delta_{\Delta x}(P^{\beta}_{\varphi_1}(s) - P^{\beta}_{\varphi_2}(s))|ds),$$

where $k_4 = k_4(T, \delta, \rho, R, \|B\|_{1+\sigma}) > 0$ does not depend on φ_1 and φ_2. Changing the variables in (5.21) and using the Gronwall lemma, similarly to (5.11) we obtain

$$|\delta_{\Delta x}(X^{\beta}_{\varphi_1}(\tau) - X^{\beta}_{\varphi_2}(\tau))| + |\delta_{\Delta x}(P^{\beta}_{\varphi_1}(\tau) - P^{\beta}_{\varphi_2}(\tau))| \le k_5(\|\varphi_1 - \varphi_2\|_{L_1((0,t), C^2(\overline{\mathbb{R}^2_+}))}|\Delta x|^{\sigma}, \tag{5.22}$$

where $k_5 = k_5(T, \delta, \rho, R, \|B\|_{1+\sigma}) > 0$ does not depend on φ_1 and φ_2.

Inequalities (5.15)–(5.17) and (5.22) imply that

$$\frac{|\delta_{\Delta x}(F_{\varphi_1}(x,t) - F_{\varphi_2}(x,t))|}{|\Delta x|^{\sigma}} \le k_6 m_{1+\sigma}\|\varphi_1 - \varphi_2\|_{L_1((0,t), C^2(\overline{\mathbb{R}^3_+}))}, \tag{5.23}$$

where $k_6 = k_6(T, \delta, \rho, R, \|B\|_{1+\sigma}) > 0$ does not depend on φ_1 and φ_2.

From (5.13), (5.23) and Remark 2.1 we obtain (5.5).

6. EXISTENCE AND UNIQUENESS OF CLASSICAL SOLUTION

6.1. First we consider the auxiliary problem

$$-\Delta u(x) = F(x) \quad (x \in \mathbb{R}^3_+), \tag{6.1}$$

$$u(x)|_{x_1=0} = 0 \quad (x' \in \mathbb{R}^2). \tag{6.2}$$

Let $C_0(\overline{\mathbb{R}^3_+}) = \{w \in C(\overline{\mathbb{R}^3_+})\colon w(x) \to 0 \text{ as } |x| \to \infty\}$.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and let $\bar{\Omega} \subset \mathbb{R}^3_+$. For any $F \in C^{\sigma}_{\Omega}(\overline{\mathbb{R}^3_+})$, there exists a unique solution $u \in C^{2+\sigma}(\overline{\mathbb{R}^3_+}) \cap C_0(\overline{\mathbb{R}^3_+})$ of problem* (6.1) (6.2), *and*

$$\|u\|_{2+\sigma} \le c_3\|F\|_{\sigma}, \tag{6.3}$$

where $c_3 = c_3(\Omega, \sigma) > 0$ does not depend on F.

For a proof, see Lemma 5.3 in [31].

6.2. Now we can formulate the main result of this paper.

Theorem 6.1. *Let Conditions* 2.1 *and* 2.2 *hold. Assume also that the following inequality is fulfilled:*

$$4\pi e c_1 c_3 m_{\sigma} < R, \tag{6.4}$$

where $c_1, c_3 > 0$ are constants from Lemmas 5.2, 6.1 *with $\Omega = \mathbb{R}^3_{7\delta/8} \cap B_{\varkappa^{-1}-\delta/8}$.*

Then there exists a unique classical solution of problem (1.1)–(1.4) *such that* $\varphi \in M_{2+\sigma,R}$ and $\operatorname{supp} f^{\beta}(\cdot,\cdot,t) \subset \mathfrak{D}^{\beta}_0$ *for all $t \in [0,T]$.*

Proof. 1. For each function $\varphi \in M_{2+\sigma,R}$, we denote by u_{φ} a classical solution of problem (6.1), (6.2) with $F = 4\pi e F_{\varphi}(x,t)$, where the function $F_{\varphi}(x,t)$ is given by (4.11). By Lemmas 4.2 and 5.1, $F_{\varphi} \in C([0,T], C^{\sigma}_{\Omega}(\overline{\mathbb{R}^3_+}))$. Therefore Lemma 6.1 implies that $u_{\varphi} \in C([0,T], C^{2+\sigma}(\overline{\mathbb{R}^3_+}))$. Denote u_{φ} by $A\varphi$.

By assumption, $4\pi e c_1 c_3 m_{\sigma} < R$. Hence, by virtue of (5.1) and (6.3), we have

$$\|A\varphi\|_{2+\sigma,T} < R \quad \text{for} \quad \varphi \in M_{2+\sigma,R}. \tag{6.5}$$

Therefore the operator A maps $M_{2+\sigma,R}$ into itself.

2. In the complete metric space $M_{2+\sigma,R}$ we introduce the equivalent metric given by

$$\rho'_{2+\sigma,R}(\varphi_1, \varphi_2) = \sup_{0<t<T}(\|\varphi_1(\cdot,t) - \varphi_2(\cdot,t)\|_{2+\sigma}\exp(-kt)),$$

where $k > 0$ is sufficiently large, $\varphi_1, \varphi_2 \in M_{2+\sigma,R}$.

From Lemmas 6.1 and 5.3 it follows that

$$\|(A\varphi_1 - A\varphi_2)(\cdot,t)\|_{2+\sigma} \le 4\pi e c_3\|(F_{\varphi_1} - F_{\varphi_2})(\cdot,t)\|_{\sigma} \le 4\pi e m_{1+\sigma} c_2 c_3\|\varphi_1 - \varphi_2\|_{L_1((0,t), C^2(\overline{\mathbb{R}^3_+}))} \tag{6.6}$$

for any $\varphi_1, \varphi_2 \in M_{2+\sigma,R}$.

Clearly,

$$\|\varphi_1-\varphi_2\|_{L_1((0,t),C^2(\overline{\mathbb{R}^3_+}))}\exp(-kt)\le\int_0^t\|(\varphi_1-\varphi_2)(\cdot,s)\|_{2+\sigma}\exp(-ks)\exp(k(s-t))ds$$
$$\le\rho'_{2+\sigma,R}(\varphi_1,\varphi_2)\int_0^t\exp(k(s-t))ds\le\frac{1}{k}\rho'_{2+\sigma,R}(\varphi_1,\varphi_2).$$

Therefore, multiplying (6.6) by $\exp(-kt)$ and taking the supremum as $t\in(0,T)$, we obtain

$$\rho'_{2+\sigma,R}(A\varphi_1,A\varphi_2)\le\frac{4\pi e m_{1+\sigma}c_2c_3}{k}\rho'_{2+\sigma,R}(\varphi_1,\varphi_2).$$

Let $k=8\pi e m_{1+\sigma}c_2c_3$. Then we have

$$\rho'_{2+\sigma,R}(A\varphi_1,A\varphi_2)\le\frac{1}{2}\rho'_{2+\sigma,R}(\varphi_1,\varphi_2). \tag{6.7}$$

From (6.5) and (6.7) it follows that the operator $A: M_{2+\sigma,R}\to M_{2+\sigma,R}$ has a unique fixed point $\varphi\in M_{2+\sigma,R}$. Thus problem (1.1)–(1.4) has a unique classical solution $\{\varphi, f_\varphi^\beta\}$, where φ is a fixed point of A and f_φ^β is given by (4.10). From Lemma 4.2 we obtain $\operatorname{supp} f_\varphi^\beta(\cdot,\cdot,t)\subset\mathcal{D}_0^\beta$ for all $t\in[0,T]$.

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